

# THE DIEDERICH-FORNÆSS EXPONENT AND NON-EXISTENCE OF STEIN DOMAINS WITH LEVI-FLAT BOUNDARIES

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**ABSTRACT.** We study the Diederich-Fornæss exponent and relate it to non-existence of Stein domains with Levi-flat boundaries in complex manifolds. In particular, we prove that if the Diederich-Fornæss exponent of a smooth bounded Stein domain in an  $n$ -dimensional complex manifold is  $> k/n$ , then it has a boundary point at which the Levi-form has rank  $\geq k$ .

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## 1. INTRODUCTION

A domain in a complex manifold is Stein if and only if there exists a smooth strictly plurisubharmonic exhaustion function. A Stein domain is called hyperconvex if there exists a smooth bounded strictly plurisubharmonic function. Diederich and Fornæss [11] showed that for any bounded pseudoconvex domain in  $\mathbb{C}^n$  with  $C^2$  boundary, there exist a positive constant  $\eta$  and a defining function  $\rho$  such that  $\hat{\rho} = -(-\rho)^\eta$  is plurisubharmonic on  $\Omega$  (see also [29]). The existence of bounded plurisubharmonic function was later generalized to bounded pseudoconvex domains with  $C^1$  boundary by Kerzman and Rosay [20] and with Lipschitz boundary by Demailly [10] (see also more recent results by Harrington [15]). The constant  $\eta$  is called a Diederich-Fornæss exponent. The supremum of all Diederich-Fornæss exponents is called the Diederich-Fornæss index of  $\Omega$ . The Diederich-Fornæss index has implications in regularity theory of the  $\bar{\partial}$ -Neumann Laplacian (see, for example, [21, 3, 5, 16]), as well as in estimates of the pluri-complex Green function [4] and comparison of the Bergman and Szegő kernels [8]. The Diederich-Fornæss indices can be arbitrarily small on the worm domains ([11, 12]). Sibony proved that for a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  satisfying Property (P), the Diederich-Fornæss index is one (see [31]).

If the pseudoconvex domain  $\Omega$  has a defining function which is bounded plurisubharmonic on  $\bar{\Omega}$ , then the Diederich-Fornæss index is one. Fornæss and Herbig [13] showed that a smooth bounded domain in  $\mathbb{C}^n$  with a defining function that is plurisubharmonic on the boundary also has Diederich-Fornæss index one. In this case, Boas and Straube showed that the  $\bar{\partial}$ -Neumann Laplacian is global hypoelliptic on  $L^2$ -Sobolev spaces (see [1]). It was shown by Nemirovskii [22, Corollary] that any smooth bounded Stein domain with a defining function that is plurisubharmonic on the domain cannot have Levi-flat boundary. In this paper, we study the Diederich-Fornæss exponent and relate it to non-existence of Stein domains with Levi-flat boundaries in complex manifolds. Our main result can be stated as follows:

**Theorem 1.1.** *Let  $\Omega$  be a bounded Stein domain with  $C^2$  boundary in a complex manifold  $M$  of dimension  $n$ . If the Diederich-Fornæss index of  $\Omega$  is greater than  $k/n$ ,  $1 \leq k \leq n-1$ , then  $\Omega$  has a boundary point at which the Levi form has rank greater than  $k$ .*

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In particular, we have the following corollary.

**Corollary 1.2.** *If the Diederich-Foræss index is greater than  $1/n$ , then its boundary cannot be Levi flat; and if the Diederich-Foræss exponent is greater than  $1 - 1/n$ , then its boundary must have at least one strongly pseudoconvex boundary point.*

We would like to thank Professor Takeo Ohsawa who kindly informed us that similar results were obtained by Adachi and Brinkschulte independently using different methods [37]. For related work on the nonexistence of Levi-flat hypersurfaces in complex manifolds, we refer to the reader to papers [18, 22, 32, 33, 23, 5, 6, 25, 26] in the references.

## 2. THE DIEDERICH-FORNÆSS INDEX

Let  $M$  be an  $n$ -dimensional complex manifold with hermitian metric  $\omega$ . Let  $\Omega$  be a bounded domain in  $M$ . A continuous real-valued function  $r$  on  $M$  is called a defining function of  $\Omega$  if  $r < 0$  on  $\Omega$ ,  $r > 0$  on  $M \setminus \overline{\Omega}$ , and  $C_1\delta(z) \leq |r(z)| \leq C_2\delta(z)$  near  $b\Omega$ , where  $\delta(z)$  is the geodesic distance from  $z$  to the boundary  $b\Omega$ . We will also assume that the defining function  $r$  is in the same smoothness class as that of the boundary  $b\Omega$ . A defining function  $r$  is said to be normalized if  $\lim_{z \rightarrow b\Omega} |r(z)|/\delta(z) = 1$ . Note that the signed distance function  $\rho(z) = -\delta(z)$  on  $\Omega$  and  $\rho(z) = \delta(z)$  on  $M \setminus \Omega$  is a normalized defining function for  $\Omega$ .

A constant  $0 < \eta \leq 1$  is called a *Diederich-Fornæss exponent* of a defining function  $r$  of  $\Omega$  if there exists a neighborhood  $U$  of  $b\Omega$  such that

$$(2.1) \quad \partial\bar{\partial}(-(-r)^\eta) \geq 0$$

on  $U \cap \Omega$  in the sense of distribution. We will call the supremum of all such  $\eta$ 's the *Diederich-Fornæss index* of  $r$  and denote it by  $I(r)$ . The supremum of  $I(r)$  over all defining functions of  $\Omega$  is called the *Diederich-Fornæss index* of  $\Omega$  and is denoted by  $I(\Omega)$ .

A defining function  $r$  is said to satisfy the *strong Oka property* if there exists a constant  $K$  and a neighborhood  $U$  of  $b\Omega$  such that

$$(2.2) \quad \partial\bar{\partial}(-\log(-r)) \geq K\omega$$

on  $U \cap \Omega$  in the sense of distribution. The supremum of all such  $K$ 's is called the *Oka index* of  $r$  and is denoted by  $K(r)$ . By Takeuchi's theorem, the signed distance function of a (proper) pseudoconvex domain in  $\mathbb{CP}^n$  with the Fubini-Study metric satisfies the strong Oka property with Oka index  $1/12$ . (Hereafter, the Fubini-Study metric is normalized so that its holomorphic sectional curvature is 2 and hence its holomorphic bisectional curvature is  $\geq 1$ .)

Let  $\Omega \subset\subset M$  be a bounded domain with  $C^2$ -boundary. Let  $r$  be a defining function of  $\Omega$ . Let  $\omega_\nu = \partial r / |\partial r|$ . Let  $L_\nu$  be the dual vector of  $\omega_\nu$ . For any  $(1,0)$ -vector  $X$  near  $b\Omega$ , let  $X_\nu = \langle X, L_\nu \rangle_\omega L_\nu$  be the complex normal component of  $X$  and  $X_\tau = X - X_\nu$  the complex tangential component. Write  $T^{1,0}(r) = \{(z, X) \in T^{1,0}(M) \mid Xr = 0\}$ . For  $z \in b\Omega$ , we further decompose  $X_\tau = X_s + X_l$ , where  $X_l$  is in the null space  $\mathcal{N}_z$  of the Levi-form  $\partial\bar{\partial}\delta$  at  $z$  and  $X_s \perp X_l$ . Let  $S^{1,0}(M) = \{(z, X) \in T^{1,0}(M), |X|_\omega = 1\}$ . Let  $W$  be the weakly pseudoconvex points on  $b\Omega$ . Let

$$S(r) = \max\{|\partial\bar{\partial}r(X_l, \bar{L}_\nu)(z)|; \quad |X_l|_\omega = 1, X_l \in \mathcal{N}_z, z \in W\}.$$

If  $b\Omega$  is strongly pseudoconvex, we set  $S(r) = 0$ . Define

$$(2.3) \quad I_0(r) = \max \left\{ \min \left\{ \frac{K(r)}{8(S(r))^2}, \frac{1}{2} \right\}, 1 - \frac{2(S(r))^2}{K(r)} \right\} > 0.$$

**Theorem 2.1.** *Let  $\Omega$  be a bounded domain with  $C^2$ -boundary and let  $r$  be a normalized defining function that satisfies the strong Oka property. Then  $I(r) \geq I_0(r)$ .*

*Proof.* A simple computation yields that

$$(2.4) \quad \partial\bar{\partial}(-\log(-r)) = \frac{\partial\bar{\partial}r}{-r} + \frac{\partial r \wedge \bar{\partial}r}{r^2}$$

and

$$(2.5) \quad \begin{aligned} \partial\bar{\partial}(-(-r)^\eta) &= \eta(-r)^\eta \left( \frac{\partial\bar{\partial}r}{-r} + (1-\eta) \frac{\partial r \wedge \bar{\partial}r}{r^2} \right) \\ &= \eta(-r)^\eta \left( \partial\bar{\partial}(-\log(-r)) - \eta \frac{\partial r \wedge \bar{\partial}r}{r^2} \right). \end{aligned}$$

It follows from (2.5) that (2.1) is equivalent to

$$(2.6) \quad \partial\bar{\partial}(-\log(-r)) \geq \eta \frac{\partial r \wedge \bar{\partial}r}{r^2}.$$

Let  $c_0$  be a constant such that  $0 < c_0 < K(r)$ . Then

$$(2.7) \quad \partial\bar{\partial}(-\log(-r)) \geq c_0\omega$$

for  $z \in \Omega$  near the boundary. It follows from (2.4) that

$$(2.8) \quad \frac{\partial\bar{\partial}r(X_\tau, \bar{X}_\tau)}{-r} \geq c_0|X_\tau|_\omega^2.$$

Let  $C_1$  be any constant such that  $C_1 > S(r)$ . Then there exists a neighborhood  $U$  of  $\mathcal{N}^{1,0}(W) = \{(z, X) \mid z \in W, X \in \mathcal{N}_z, |X|_\omega = 1\}$  in  $S^{1,0}(M)$  such that

$$(2.9) \quad |\partial\bar{\partial}r(X, \bar{L}_\nu)| \leq C_1, \quad (z, X) \in U.$$

For  $(z, X_\tau) \in S^{1,0}(\bar{\Omega}) \setminus U$  with  $z$  near  $b\Omega$ ,

$$(2.10) \quad \partial\bar{\partial}r(X_\tau, \bar{X}_\tau) \geq C_2|X_\tau|_\omega^2$$

for some constant  $C_2 > 0$ . We write  $X = X_\tau + X_\nu$  with  $X_l \in \mathcal{N}_z$  as before. Then

$$(2.11) \quad \begin{aligned} \partial\bar{\partial}(-\log(-r))(X, \bar{X}) &= \frac{\partial\bar{\partial}r(X_\tau, \bar{X}_\tau)}{-r} + \frac{\partial\bar{\partial}r(X_\nu, \bar{X}_\nu)}{-r} \\ &\quad + \frac{2 \operatorname{Re} \partial\bar{\partial}r(X_\tau, \bar{X}_\nu)}{-r} + \frac{|Xr|^2}{r^2}. \end{aligned}$$

Note that  $|Xr| = |X_\nu|_\omega \cdot |\partial r|_\omega$ . Let  $K_0 = \sup\{|\partial\bar{\partial}r|_\omega; z \in \bar{\Omega}\}$ . Then

$$(2.12) \quad |\partial\bar{\partial}r(X_\nu, X_\nu)| \leq K_0|Xr|^2/|\partial r|_\omega^2$$

Similarly,

$$(2.13) \quad |\operatorname{Re} \partial\bar{\partial}r(X_\tau, \bar{X}_\nu)| \leq K_0|X_\tau|_\omega \cdot |Xr|/|\partial r|_\omega.$$

We first deal with the strictly pseudoconvex directions. For  $(z, X) \in T^{1,0}(\Omega)$  with  $(z, X_\tau/|X_\tau|) \in S^{1,0}(\Omega) \setminus U$  with  $z$  near  $b\Omega$ , it follows from (2.13) and (2.10) that for any positive constant  $M$ ,

$$(2.14) \quad \begin{aligned} |2 \operatorname{Re} \partial\bar{\partial}r(X_\tau, \bar{X}_\nu)| &\leq K_0 \left( \frac{1}{M}|X_\tau|_\omega^2 + \frac{M}{|\partial r|_\omega^2}|Xr|^2 \right) \\ &\leq \frac{K_0}{MC_2} \partial\bar{\partial}r(X_\tau, \bar{X}_\tau) + \frac{K_0 M}{|\partial r|_\omega^2} |Xr|^2. \end{aligned}$$

Therefore,

$$(2.15) \quad \begin{aligned} \partial\bar{\partial}(-\log(-r))(X, \bar{X}) &\geq \left(1 - \frac{K_2}{MC_2}\right) \frac{\partial\bar{\partial}r(X_\tau, \bar{X}_\tau)}{-r} \\ &\quad + \left(1 - \frac{K_0(M+1)|r|}{|\partial r|_\omega^2}\right) \frac{|Xr|^2}{r^2}. \end{aligned}$$

By choosing  $M$  sufficiently large and then letting  $z$  be sufficiently close to  $b\Omega$ , we know that (2.6) holds for any  $\eta < 1$ .

We now deal with weakly pseudoconvex directions. For  $(z, X) \in T^{1,0}(\Omega)$  with  $(z, X_\tau/|X_\tau|_\omega) \in U$ , we have

$$(2.16) \quad 2|\partial\bar{\partial}r(X_\tau, \bar{X}_\nu)| \leq 2C_1|X_\tau|_\omega|Xr|/|\partial r|_\omega \leq C_1\left(\frac{|r|}{\varepsilon}|X_\tau|_\omega^2 + \frac{\varepsilon}{|r|}\frac{|Xr|^2}{|\partial r|_\omega^2}\right),$$

where  $\varepsilon$  is a positive constant to be chosen. Since  $r$  is a normalized defining function,  $|\partial r|_\omega = 1/\sqrt{2}$  on  $b\Omega$ . Combining (2.16) with (2.8), we have

$$(2.17) \quad \begin{aligned} \partial\bar{\partial}(-\log(-r))(X, \bar{X}) &\geq (c_0 - C_1/\varepsilon)|X_\tau|^2 + \frac{1 - (C_1\varepsilon + K|r|)|\partial r|_\omega^{-2}}{r^2}|Xr|^2 \\ &\geq (c_0 - C_1/\varepsilon)|X_\tau|^2 + \frac{1 - 2C_1\varepsilon - K'|r|}{r^2}|Xr|^2 \end{aligned}$$

for some positive constant  $K'$ .

We consider two cases:  $4C_1^2 \leq c_0$  and  $4C_1^2 > c_0$ . When  $4C_1^2 \leq c_0$ , we take  $\varepsilon = C_1/c_0$ . Then

$$(2.18) \quad \partial\bar{\partial}(-\log(-r))(X, \bar{X}) \geq (1 - 2C_1^2/c_0 - K'|r|)|Xr|^2/r^2.$$

When  $4C_1^2 > c_0$ , we take  $\varepsilon = \frac{1}{4C_1} < C_1/c_0$ . Then combining (2.17) with (2.7), we have

$$\partial\bar{\partial}(-\log(-r))(X, \bar{X}) \geq -\left(\frac{C_1}{c_0\varepsilon} - 1\right)\partial\bar{\partial}(-\log(-r))(X, \bar{X}) + \frac{1 - 2C_1\varepsilon - K'|r|}{r^2}|Xr|^2.$$

Therefore,

$$\partial\bar{\partial}(-\log(-r))(X, \bar{X}) \geq \left(\frac{c_0\varepsilon(1 - 2C_1\varepsilon)}{C_1} - \frac{K'c_0\varepsilon|r|}{C_1}\right) \frac{|Xr|^2}{r^2}.$$

Hence

$$(2.19) \quad \partial\bar{\partial}(-\log(-r))(X, \bar{X}) \geq \left(\frac{c_0}{8C_1^2} - \frac{K'c_0\varepsilon|r|}{2C_1^2}\right) \frac{|X_\nu|^2}{r^2}.$$

Note that when  $4C_1^2 \leq c_0$ , we have

$$(2.20) \quad 1 - \frac{C_1^2}{c_0} \geq \frac{1}{2} \quad \text{and} \quad \frac{c_0}{4C_1^2} \geq \frac{1}{2}.$$

Furthermore, when  $4C_1^2 > c_0$ ,

$$(2.21) \quad \frac{1}{2} > \frac{c_0}{8C_1^2} > 1 - \frac{C_1^2}{c_0}.$$

Combing (2.18)-(2.21), we know that (2.6) holds for any  $\eta < I_0(r)$ . We thus conclude the proof of Proposition 2.1  $\square$

By Takeuchi's theorem ([36], see also [7, 14]), (2.2) holds for the signed distance function with  $K = 1/12$  on any proper pseudoconvex domain on complex projective space  $\mathbb{CP}^n$ . Combing this with Proposition 2.1, we have:

**Corollary 2.2.** *Let  $\Omega$  be a proper pseudoconvex domain in  $\mathbb{CP}^n$  with  $C^2$  boundary. Then its Diederich-Fornaess index*

$$I(\Omega) \geq I_0(\rho) = \max \left\{ \min \left\{ \frac{1}{96(S(\rho))^2}, \frac{1}{2} \right\}, 1 - 24(S(\rho))^2 \right\} > 0,$$

where  $\rho$  is the signed distance function to  $b\Omega$  with respect to the Fubini-Study metric.

**Proposition 2.3.** *Let  $\Omega \subset\subset M$  be a bounded domain with  $C^2$  boundary and let  $r$  be a normalized defining function. Suppose (2.2) holds and there exist a neighborhood  $V$  of the set  $W$  of weakly pseudoconvex boundary points and a positive constant  $K_1 > 1$  such that*

$$(2.22) \quad K|X_\tau|^2 \leq \frac{\partial \bar{\partial} r(X_\tau, \bar{X}_\tau)}{r} \leq K K_1 |X_\tau|^2,$$

for all  $z \in V$  and  $X \in T_z^{1,0}(M)$ . Then

$$(2.23) \quad I(\Omega) \geq \max \left\{ \min \left\{ \frac{1}{8(K_1 - 1)}, \frac{1}{2} \right\}, 3 - 2K_1 \right\}.$$

*Proof.* From (2.2), we know that

$$\Theta = \partial \bar{\partial}(-\log(-r)) - K\omega$$

is positive semi-definite. Applying the Cauchy-Schwarz inequality to  $\Theta(X_\tau, X_\nu)$ , we then have

$$|\Theta(X_\tau, X_\nu)| \leq |\Theta(X_\tau, X_\tau)|^{1/2} |\Theta(X_\nu, X_\nu)|^{1/2}.$$

(We refer the reader to [34] for a similar technique that has been used by Straube to construct Stein neighborhood bases in connection with regularity theory in the  $\bar{\partial}$ -Neumann problem.) Therefore,

$$\left| \frac{\partial \bar{\partial} r(X_\tau, \bar{X}_\nu)}{r} \right|^2 \leq \left( \frac{\partial \bar{\partial} r(X_\tau, X_\tau)}{-r} - K|X_\tau|_\omega^2 \right) \left( \frac{\partial \bar{\partial} r(X_\nu, X_\nu)}{-r} + \frac{|X_\nu|^2}{r^2} - K|X_\nu|_\omega^2 \right).$$

Thus

$$|\partial \bar{\partial} r(X_\tau, \bar{X}_\nu)| \leq ((K_1 - 1)K)^{1/2} (1 + C|r|)^{1/2} |X_\tau| |X_\nu|,$$

for some positive constant  $C$ . The inequality (2.23) then follows by applying Proposition 2.1 with  $S(r) = ((K_1 - 1)K)^{1/2}$ .  $\square$

Let  $f \in C^2(M)$ . Recall that the real Hessian  $H_f$  is defined by

$$H_f(\xi, \zeta)(z) = \langle \nabla_\xi(\nabla f), \zeta \rangle$$

for  $\xi, \zeta \in T_{\mathbb{R}}(M^{2n})$ , where  $\nabla_\xi$  denotes the covariant derivative. For any  $X \in T_{\mathbb{C}}^{1,0}(M)$ , we write  $X = \frac{1}{\sqrt{2}}(\xi_X - \sqrt{-1}J\xi_X)$  where  $J$  is the complex structure. Let  $z$  be a point in  $\Omega$  near the boundary and  $\pi(z)$  be its closest point on  $b\Omega$ . Let  $\gamma(t)$  be the geodesic parametrized by arc-length such that  $\gamma(0) = \pi(z)$ . For any  $(1, 0)$  tangent vector  $X$  at  $z$  near  $b\Omega$ , we let  $X(t)$  be the vector at  $\gamma(t)$  obtained by parallel translate (of real and imaginary parts) of  $X$  along the geodesic from  $z$  to  $\gamma(t)$  and let  $X^0 = X(0)$ .

**Proposition 2.4.** *Let  $\Omega$  be a proper pseudoconvex domain with  $C^2$  boundary in  $\mathbb{CP}^n$ . Let  $\rho$  be the signed distance function to  $b\Omega$  with respect to the Fubini-Study metric. Let*

$$K_2 = \max\{|\nabla_{\xi_X}(\nabla \rho)|_\omega^2 + |\nabla_{J\xi_X}(\nabla \rho)|_\omega^2; z \in W, X \in \mathcal{N}_z, |X|_\omega = 1\}.$$

Then

$$I(\Omega) \geq \max \left\{ \min \left\{ \frac{1}{8(K_2 - 1)}, \frac{1}{2} \right\}, 3 - 2K_2 \right\}.$$

*Proof.* It follows from the computations in [35] that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \left( \partial \bar{\partial} \rho(X_\tau(t), X_\tau(t)) - \partial \bar{\partial} \rho(X^0, \bar{X}^0) \right) = |\nabla_{\xi_X}(\nabla \rho)|_\omega^2 + |\nabla_{J\xi_X}(\nabla \rho)|_\omega^2$$

(The above identity was proved in [35] for  $\Omega$  in  $\mathbb{C}^n$ ; compare also [34]. The proof for  $\Omega$  in  $\mathbb{CP}^n$  is similar; see [7] for related arguments.) We then conclude the proof by applying Proposition 2.3 with  $K = 1$  and any  $K_1 > K_2$ .  $\square$

From Proposition 2.1, we also obtain the following slight variation of a result of Ohsawa and Sibony ([27]; see also [5, 6]):

**Corollary 2.5.** *Let  $\Omega$  be a bounded domain in  $M$  with  $C^2$  boundary. Suppose  $r$  is a normalized defining function that satisfies (2.2). Then for any  $c \in (0, K)$  and  $\eta \in (0, I_0(r))$ , there exists a neighborhood  $V$  of  $b\Omega$  such that*

$$\partial \bar{\partial}(-\log(-r)) \geq c\omega + (1 - \frac{c}{K})\eta \frac{\partial r \wedge \bar{\partial} r}{r^2}$$

and

$$\partial \bar{\partial}(-(-r)^\eta) \geq \eta(-r)^\eta \left( c\omega + (1 - \frac{c}{K})\eta \frac{\partial r \wedge \bar{\partial} r}{r^2} \right).$$

### 3. NON-EXISTENCE OF STEIN DOMAINS WITH LEVI-FLAT BOUNDARIES

We prove Theorem 1.1 in this section. We first recall the following well-known simple lemma. Let  $\Omega$  be a bounded domain with  $C^2$  boundary in a complex hermitian manifold  $M$  of dimension  $n$ . Let  $\rho$  be a defining function for  $\Omega$ . For  $t > 0$ , let  $\Omega_{-t} = \{z \in \Omega; \rho < -t\}$ . Let  $i_t: b\Omega_{-t} \rightarrow M$  be the inclusion map. Let  $1 \leq k \leq n$  be an integer.

**Lemma 3.1.** *If the rank of the Levi form of  $b\Omega$  is  $\leq k - 1$  at all  $z \in b\Omega$ , then*

$$(3.1) \quad i_t^*(d^c \rho \wedge (dd^c \rho)^{n-1}) = O(t^{n-k}) dS_t$$

where  $dS_t$  is the surface element of  $b\Omega_{-t}$ .

We sketch the proof for the reader's convenience. Note that  $dS_t = i_t^*(\ast dr)/|dr|_\omega$  and

$$i_t^*(d^c \rho \wedge (dd^c \rho)^{n-1}) = \nu \lrcorner ((d\rho/|d\rho|) \wedge d^c \rho \wedge (dd^c \rho)^{n-1})$$

where  $\nu$  is the dual vector of  $d\rho/|d\rho|_\omega$ . By choosing local holomorphic coordinates that diagonalize the Levi form, we then obtain (3.1).

We now prove Theorem 1.1. Let  $\rho$  be a defining function of  $\Omega$  such that  $\hat{\rho} = -(-\rho)^\eta$  is plurisubharmonic on  $\Omega$  for some constant  $\eta > k/n$ . Let  $\Omega_{-t} = \{\rho < -t\}$ ,  $t > 0$ . Since  $\Omega$  is Stein,  $\Omega_{-t}$  has at least a strictly pseudoconvex boundary point for sufficiently small  $t$ . Let

$$f(t) = \int_{\Omega_{-t}} (dd^c \hat{\rho})^n.$$

Then  $f(t) \geq 0$  and  $f(t)$  is decreasing. By Stokes' theorem,

$$f(t) = \int_{b\Omega_{-t}} i_t^*(d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1}).$$

Since

$$d^c \hat{\rho} = i\eta(-\rho)^{\eta-1}(\bar{\partial}\rho - \partial\rho) \quad \text{and} \quad dd^c \hat{\rho} = 2i\eta\rho^\eta \left( \frac{\partial \bar{\partial} \rho}{-\rho} + (1 - \eta) \frac{\partial \rho \wedge \bar{\partial} \rho}{\rho^2} \right),$$

we have

$$d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1} = \eta^n (-\rho)^{n(\eta-1)} d^c \rho \wedge (dd^c \rho)^{n-1}.$$

Suppose the Levi rank of  $b\Omega$  is  $\leq k-1$  at all boundary points, then by Lemma 3.1,

$$i_t^*(d^c \rho \wedge (dd^c \rho)^{n-1}) = O(t^{n-k}) dS_t.$$

Thus

$$f(t) = O(t^{n\eta-k}).$$

Therefore,  $\lim_{t \rightarrow 0^+} f(t) = 0$  and hence  $f(t) = 0$  for small  $t > 0$ . This implies that  $b\Omega_{-t}$  has Levi rank  $\leq n-2$  at each point, which leads to a contradiction. This concludes the proof of Theorem 1.1.

Corollary 1.2 follows easily. The following theorem is a variation of Theorem 1.1.

**Theorem 3.2.** *Let  $M$  be a complex manifold of dimension  $n$  with a hermitian metric  $\omega$ . Let  $\Omega$  be a bounded Stein domain in  $M$  with  $C^2$  boundary. Suppose there exist a defining function  $\rho$ , a constant  $\eta > 0$ , and a neighborhood  $U$  of  $b\Omega$  such that*

$$(3.2) \quad \partial \bar{\partial}(-(-\rho)^\eta) \geq c(-\rho)^\eta \left( \omega + \frac{\partial \rho \wedge \bar{\partial} \rho}{\rho^2} \right)$$

*on  $U \cap \Omega$  for some constant  $c > 0$ . If  $\eta \geq 1/n$ , then  $\Omega$  cannot have Levi-flat boundary.*

*Proof.* In light of Theorem 1.1, it remains to prove the case when  $\eta = 1/n$ . We follow the notations as in the above proof of Theorem 1.1. Let  $\varepsilon_0$  be sufficiently small such that  $\Omega \setminus \Omega_{-\varepsilon_0} \subset U \cap \Omega$ . We set

$$f(t) = \int_{\Omega_{-t} \setminus \Omega_{-\varepsilon_0}} (dd^c \hat{\rho})^n$$

for  $0 < t < \varepsilon_0$ . Suppose  $b\Omega$  is Levi-flat, then as in the proof of Theorem 1.1,

$$d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1} \Big|_{b\Omega_{-t}} = \eta^n (-\rho)^{n(\eta-1)} d^c \rho \wedge (dd^c \rho)^{n-1} \Big|_{b\Omega_{-t}} = O(t^{n\eta-1}) dS_t \leq C dS_t.$$

By Stoke's theorem

$$(3.3) \quad f(t) = \int_{b\Omega_{-t}} d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1} - \int_{b\Omega_{-\varepsilon_0}} d^c \hat{\rho} \wedge (dd^c \hat{\rho})^{n-1} \leq C.$$

On the other hand, it follows from (3.2) that

$$(dd^c \hat{\rho})^n \geq C \delta^{n\eta} \left( \omega + \frac{\partial \delta \wedge \bar{\partial} \delta}{\delta^2} \right)^n \geq C \delta^{n\eta-2} dV,$$

where  $dV$  is the volume element. Thus

$$\begin{aligned} f(t) &= \int_{\Omega_{-t} \setminus \Omega_{-\varepsilon_0}} (dd^c \hat{\rho})^n \geq C \int_{\Omega_{-t} \setminus \Omega_{-\varepsilon_0}} (-\rho)^{n\eta-2} dV \\ &\geq C \int_{-\varepsilon_0}^{-t} (-\rho)^{-1} d\rho \geq C(-\log t + \log \varepsilon_0). \end{aligned}$$

Therefore,  $\lim_{t \rightarrow 0^+} f(t) = \infty$ , which leads to a contradiction with (3.3). This concludes the proof of Proposition 3.2.  $\square$

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